A METHOD FOR THE SOLUTION OF NONLINEAR HEAT- AND MASS-TRANSFER EQUATIONS ON NETWORK AND COMBINED ELECTRICAL MODELS

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A term containing the transfer coefficient that is a function of the potential is isolated in the transfer equations. In mathematical simulation on electrical network models this method makes it possible to reduce the number of nonlinear elements and significantly to simplify the method of solving the nonlinear transfer equations on combined electrical models.

In the electrical simulation of solutions for nonlinear equations of energy and mass transfer on electrical network models it is necessary during the course of the solution to change the electrical resistances simulating the thermal resistances (R_λ) of the thermal conductivity to take into consideration the relationship between the transfer coefficients and the potential. To simplify the calculation we will examine the nonlinear equation of nonsteady heat conduction, written in a rectangular coordinate system:

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i}\right) - c\gamma \frac{\partial T}{\partial t} + w = 0, \tag{1}$$

although all that follows is applicable to the heat-conduction equation in any coordinate system as well as for a system of nonlinear equations of energy and mass transfer [1].

The nodes of the resistance networks (R networks) or resistance and capacitance networks (R-C networks) may be situated at the corners or within the space. The form of notation for the expression to calculate the parameters, for example, of the R networks, does not change in this case [2]. If the nodes of the R networks are situated within the space,

$$R_\lambda = h_{jk} R_N \left(\lambda \prod_{\substack{i=1 \\ i\neq j}}^{2} \sum_{k=1}^{2} h_{ik} \right)^{-1}, \tag{2}$$

$$R_t = \Delta t R_N \left(c\gamma \prod_{i=1}^{3} \sum_{k=1}^{2} h_{ik} \right)^{-1}, \tag{3}$$

$$R_w = (V_M - V) K R_N \left(w \prod_{i=1}^{3} \sum_{k=1}^{2} h_{ik} \right)^{-1}, \tag{4}$$

where h_{ik} is the distance from the node to the boundary of the volume element. If the nodes of the R network are situated at the corners,

$$R_\lambda = h_{jk} R_N \left(2\lambda \prod_{\substack{i=1 \\ i\neq j}}^{2} \sum_{k=1}^{2} h_{ik} \right)^{-1}, \tag{5}$$

where h_{ik} is the distance between the network nodes. R_t and R_w are calculated from (2) and (3), but as in the case of (4), in this case h_{ik} is the distance between the nodes.

In solving problems of R-C networks we calculate the resistances R_λ from (1) or (5), and the magnitudes of the capacitances and the values for the currents respectively simulating the thermal capacities and heat sources are determined in a corresponding way.

In solving problems with $\lambda = \lambda(T)$ on combined models [3] where the resistances R_λ are made of an electrically conductive continuous medium, the achievement of the condition $\lambda = \lambda(T)$, particularly in the solution of nonsteady problems, complicates the solution to such an extent that the combined models are used in practical terms only when λ = const. Below we present a method which makes it possible in the solution of nonlinear problems when $\lambda = \lambda(T)$ to use constant resistances for R_λ in R and R-C networks in the place of variable resistances, or to use plates of constant electrical conductivity in combined models.

For the one-dimensional problem, Eq. (1) has the form

$$\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) - c\gamma \frac{\partial T}{\partial t} + w = 0. \tag{6}$$

We write (6) in the form

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{\lambda}\frac{\partial \lambda}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 - \frac{1}{a}\frac{\partial T}{\partial t} + \frac{w}{\lambda} = 0, \tag{7}$$

where $a = \lambda/c\gamma$; λ, c, and γ, just as w, are functions of T. The expressions for the calculation of the R-network parameters, proceeding from (7), are derived in the same manner as in [2], and have the form

$$R'_\lambda = h_i R_N, \tag{8}$$

$$R'_t = a \Delta t R_N \left(\sum_{i=1}^{2} h_i \right)^{-1}, \tag{9}$$

$$R'_w = (V_M - V)\lambda K R_N \times$$
$$\times \left\{ \left[w + \frac{\partial \lambda}{\partial T}\left(\frac{\partial T}{\partial x}\right)^2 \right] \sum_{i=1}^{2} h_i \right\}^{-1}. \tag{10}$$

Here h_i is the distance from the node to the boundaries of the segment, since (8)–(10) are written for the case in which the nodes are situated within the element. The expressions for R'_λ, R'_t, and R'_w are written analogously if the nodes are situated at the corners.

The body resistances R_λ are independent of λ, while the magnitudes of the variable resistances R'_w, in this approach to the solution of the nonlinear problems depend at each step on the values of $\lambda(T)$, w(T), $\partial\lambda/\partial T$, and $\partial T/\partial x$. If $\partial\lambda/\partial T$ is not given analytically as a function of T, proceeding from a graph or table for $\lambda(T)$, we find

$$\frac{\partial\lambda}{\partial T}=\frac{\lambda_{T_{n-1}}-\lambda_{T_{n+1}}}{T_{n-1}-T_{n+1}}. \quad (11)$$

The more exact approximation of the broken line for the function $\lambda(T)$, the more exact the determination from (11) of $\partial\lambda/\partial T$ for each value of the temperature $T_n = (T_{n-1} + T_{n+1})/2$.

The derivative $\partial T/\partial x$ is defined as

$$\frac{\partial T}{\partial x}=(T_{i-1}-T_{i+1})\left(\sum_{k=1}^{2}h_k\right)^{-1}. \quad (12)$$

To determine the values of λ, w, $\partial\lambda/\partial T$, and $\partial T/\partial x$ we must have the temperature field of the previous approximation. Then proposed method does not differ in this from the conventional method of solving nonlinear problems on R networks.

For one-dimensional problems in which $\lambda = \lambda(x)$, all other conditions being equal, we find from (6)

$$\frac{\partial^2 T}{\partial x^2}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial x}\frac{\partial T}{\partial x}-\frac{1}{a}\frac{\partial T}{\partial t}+\frac{w}{\lambda}=0. \quad (13)$$

Then R'_λ and R'_t will have the form of (7) and (8), while

$$R'_w=(V_M-V)\lambda KR_N\left\{\left[w+\frac{\partial\lambda}{\partial x}\frac{\partial T}{\partial x}\right]\sum_{i=1}^{2}h_i\right\}^{-1}. \quad (14)$$

On a network of constant resistances for R_λ or on a combined model this approach makes it possible to solve one-dimensional problems in which $\lambda = \lambda(x)$; c, γ, and w are functions of x and T.

When $\lambda = \lambda(T)$, just as (8)–(10) were derived, proceeding from (1), it is possible to derive the parameters of the networks solving the three-dimensional problems. For R networks with nodes within the volume

$$R'_\lambda=h_{jk}R_N\left(\prod_{\substack{i=1\\ i\neq j}}^{2}\sum_{k=1}^{2}h_{ik}\right)^{-1}, \quad (15)$$

$$R'_t=a\,\Delta tR_N\left(\prod_{i=1}^{3}\sum_{k=1}^{2}h_{ik}\right)^{-1}, \quad (16)$$

$$R'_w=(V_M-V)\lambda KR_N\times$$

$$\times\left\{\left[w+\frac{\partial\lambda}{\partial T}\sum_{i=1}^{3}\left(\frac{\partial T}{\partial x_i}\right)^2\right]\prod_{i=1}^{3}\sum_{k=1}^{2}h_{ik}\right\}^{-1}. \quad (17)$$

We denote in (17) the expression in brackets by w'. For R-C networks (nodes within the volume) R'_λ is defined from (15)

$$I'=w'\prod_{i=1}^{3}\sum_{k=1}^{2}h_{ik}(\lambda KR_N)^{-1}, \quad (18)$$

$$C'=a\prod_{i=1}^{3}\sum_{k=1}^{2}h_{ik}\tau(R_N)^{-1}, \quad (19)$$

where τ is the time scale.

The method of calculating resistances for the solution of problems on combined models is covered in [3, 4]. The resistances, current, and capacitances for combined models are calculated according to expressions analogous to (15)–(19), with the difference that the quantity R_N will be a function of the specific resistance of the conducting plate (or the electrolyte layer). If there are not heat sources in (1), w in (10), (14), (17), and (18) should be assumed to be equal to zero. The total number of resistances in the solution of problems on R networks will then increase to a number equal to the number of network nodes, but the number of variable resistances required for the solution of a nonlinear problem in comparison with the case in which the resistances have been calculated according to (2)–(5) is reduced. In the solution of nonlinear problems on R-C networks according to the method described here for w = 0 additional currents will be applied to the nodes, with the magnitude of these currents determined from (18), but all of the resistances R_λ will be constant during the solution process. The advantages of this method in the solution of nonlinear transfer problems when w = 0 are obvious for combined models as well, despite the fact that additional resistances are necessary for solutions by the Libmann method (R) or additional currents in the case of solutions by the Boiken method (C).

In the solution of nonsteady problems it is possible to avoid the introduction of additional resistances or currents into the combined models when w = 0, if expressions of the type $(1/\lambda)(\partial T/\partial\lambda)(\partial T/\partial x)^2$ are introduced into (16) or into (19) as follows:

$$R''_t=\Delta tR_N\left\{\left[\frac{1}{a}+\frac{\sum_{i=1}^{3}\frac{\partial T}{\partial\lambda}\left(\frac{\partial T}{\partial x}\right)^2}{\lambda\frac{\partial T}{\partial t}}\right]\prod_{i=1}^{3}\sum_{k=1}^{2}h_{ik}\right\}^{-1}, \quad (20)$$

$$C''=\prod_{i=1}^{3}\sum_{k=1}^{2}h_{ik}\tau\left\{\left[\frac{1}{a}+\frac{\sum_{i=1}^{3}\frac{\partial T}{\partial\lambda}\left(\frac{\partial T}{\partial x}\right)^2}{\lambda\frac{\partial T}{\partial t}}\right]R_N\right\}^{-1}. \quad (21)$$

In this case $\partial T/\partial t$ should be substituted into (20) and (21), proceeding from the functions T = T(t) for the given node, derived in the previous approximations. In all earlier cases the calculation of the resistances simulating the boundary conditions is carried out in the manner of [2–4]. The derivation of the parameters for the networks or for the combined models in no way differs from the case presented above, in which the solution of a system of nonlinear equations of energy and mass transfer is simulated [1]. The general approach to the solution of such a system, e.g., in the case of the transfer of heat and moisture in bulk grain, is given in [5], but the equations should be modified in the manner of (1) or (6) above.

A one-dimensional nonlinear steady problem with an exact analytical solution was solved according to our method on an electrical model of an R network:

The problem is formulated as follows:

$$\frac{\partial}{\partial x}\left(\lambda\frac{\partial T}{\partial x}\right)=0 \quad (0<x<\delta),$$

$$T(0)=T_1,\quad T(\delta)=T_2;\quad \lambda=\lambda_0(1+bT).$$

The solution has the form

$$T=\sqrt{\left(\frac{1}{b}+T_1\right)^2-\frac{2qx}{\lambda_0 b}}-\frac{1}{b}, \tag{22}$$

where

$$q=\frac{\lambda_{av}}{\delta}(T_1-T_2); \quad \lambda_{av}=\frac{1}{T_1-T_2}\int_{T_2}^{T_1}\lambda(t)\,dt.$$

In our example δ = 0.2 m, T_1 = 625° C, T_2 = 0° C, λ_0 = 0.1, b = 0.0016.

The R network for a plate divided into eight segments consisted of constant (R_λ) and variable (R'_w) resistances of class 0.2. The measuring circuit was provided by a series-production integrator of the EGDA-9/60 type.

The first approximation (R'_w off) yields the linear distribution of temperature in the plate, i.e., the solution with λ = const.

In second approximation the R'_w calculated according to (10) for w = 0, are connected to the nodes of the model; the maximum voltage V_m is applied to the ends of R'_w.

The results of the first approximation made it possible to evaluate the total error in the measurement. Multiple repetitions of the experiment demonstrated that in our example this error did not exceed ±0.05% of the maximum voltage value, which amounted to ±0.3°. The results of the second approximation made it possible to ascertain the error in this method in comparison with the exact analytical solution according to (22). After the second approximation the maximum error did not exceed ±1.4%, i.e., ±8.8°.

After the second approximation correction factors were introduced for R'_w and the third approximation demonstrated that the maximum error in comparison with the solution according to (22) did not exceed ±0.1%, i.e., ±0.6°.

We know that even the second approximation yields satisfactory accuracy, while the third approximation makes it possible to achieve a solution for the nonlinear problem with an accuracy that is close to the accuracy of the measuring devices in the series-produced integrators of the EGDA, EI, MSM type, etc.

The proposed method may be used for the solution of nonlinear problems not only on R and R-C networks and on combined electrical models, but also in the solution of nonlinear problems on electrical models with a distributed capacitance [6], on hydraulic integrators of various designs, as well as in the solution of (1) by analytical or numerical methods.

It is obvious that the greatest advantages of the proposed method arise in the solution of nonlinear problems on combined models.

NOTATION

R_λ, R_t, and R_w are electrical resistances, with the help of which it is possible to simulate thermal resistances of thermal conductivity, heat capacity and heat sources, respectively; R_N is the scale of transition from thermal to electrical resistances; T is the temperature; t is the time; λ, c, and γ are the thermal conductivity, specific heat capacity by weight, and specific weight, respectively; w is the source (sink) of heat; V_m is the maximum (minimum) value of voltage; V is the voltage in a given assembly at a given time; K is the scale of transition from temperatures to voltages; I and C are the current and capacity simulating a heat flux w and heat capacity of an element, respectively.

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